Final mark: 8

3.5/3.5

1a $F(f+\lambda g) = \int_0^\pi (\sin t + \cos t)(f+\lambda g)(t)\, dt$

$$= \int_0^\pi (\sin t + \cos t)\,(f(t) + \lambda g(t))\, dt$$

$$= \int_0^\pi (\sin t + \cos t)\, f(t)\, dt + \lambda \int_0^\pi (\sin t + \cos t)\, g(t)\, dt = F(f) + \lambda F(g)$$

for $f, g \in L^2[0,\pi]$, $\lambda \in \mathbb{C}$

$$F(f) = \int_0^\pi (\sin t + \cos t)\, f(t)\, dt = \langle f(t), \sin t + \cos t \rangle$$

$\Rightarrow$

$$\|F\| = \|\sin t + \cos t\| = \left(\int_0^\pi |\sin t + \cos t|^2\, dt \right)^{1/2}$$

$$\int_0^\pi |\sin t + \cos t|^2\, dt$$

$$= \int_0^\pi (\sin^2 t + 2\sin t\cos t + \cos^2 t)\, dt$$

$$= \int_0^\pi (1 + 2\sin t \cos t)\, dt$$

$$= \pi + \int_0^\pi 2\sin t\cos t\, dt$$

$$= \pi + \left[\sin^2 t\right]_0^\pi$$

$$= \pi + 0 = \pi$$

$$\|F\| = \sqrt{\pi}$$

b $\exists!\ y \in L^2[0,\pi]$ such that $G(f) = \langle f, y \rangle = \int_0^\pi f(t)\bar{y}\, dt$

now take $\bar{y} = 2\pi i e^{it} g(t) \Rightarrow g(t) = \frac{1}{2\pi i e^{it}} \bar{y}$,

then $G(f) = 2\pi i \int_0^\pi e^{it} f(t) g(t)\, dt$ for $f \in L^2[0,\pi]$

So such a $g \in L^2[0,\pi]$ exists.

$$\left(\int_0^\pi |g(t)|^2\, dt = \int_0^\pi \left|\frac{1}{2\pi i e^{it}}\bar{y}\right|^2 dt \le \int_0^\pi |y(t)|^2\, dt < \infty \right)$$

$\frac{4,5}{4,5}$ 4 $\ell(x) = i + 2$

$|\ell(x)| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$\mu x \in U \Rightarrow |\ell(\mu x)| = |\mu \ell(x)| = |\mu||\ell(x)| = \sqrt{5}\,|\mu|$

$\|\mu x\| = |\mu|\,\|x\| = |\mu|$

$\Rightarrow \quad \cancel{\not} \; |\ell(\mu x)| = \sqrt{5}\,\|\mu x\|$

$\|\ell\| = \sqrt{5}$

$\Rightarrow \exists\, L \in E'$ such that $L|_U = \ell$ and $\|L\| = \|\ell\| = \sqrt{5}$

a no

b yes ✓

c no 4

$\frac{1,5}{2}$ 2 $T : \mathcal{B} \to \mathcal{B}$ bounded, linear

$\mathcal{B}$ Banach, $\operatorname{dom} T \subset \mathcal{B}$

~~dom T is open $\underset{(T\text{ lin.})}{\Rightarrow}$ $T^{-1}(\operatorname{dom} T) = T$ is open~~

~~This implies that T closed ⇒ dom T closed~~
~~also dom T closed ⇒ T closed~~
~~X normed space, U ⊂ X complete ⇒ U closed~~
~~B Banach ⇒ T complete~~
~~B normed spaces, dto~~

$\operatorname{dom} T \xmapsto{T} O$, O open $\underset{(T\text{ lin.})}{\Rightarrow}$ $\operatorname{dom} T$ is open

$\Updownarrow$

T open

cont.

You do not know if your T is open or closed!

So: dom T closed ⇒ T closed

why?

T closed ⇒ T ~~compact~~ complete ⇒ dom T ~~closed~~ complete ⇒ dom T closed (B Banach)

here you need that T is cont.

(2) So we have dom T closed $\Leftarrow$ T closed

$\frac{1}{2}$ P.

3 H Hilbert, T, S linear operators

$$(Tf, g) = (f, Sg) \qquad f, g \in H$$

$$(Tf, g) = T(f, g) \cancel{= (f, T^* g)}$$

$$T(\overline{\bar{g}, f}) = (\overline{T^* g, f}) \cancel{=}$$

$$= (f, T^* g) = (f, Sg)$$

$\Rightarrow T^* = S$ ✓

graph $T \subset H \times H$ complete ~~...~~ why?

$\Rightarrow$

graph T closed $\Rightarrow$ T closed $\Rightarrow$ T bounded

same for S still to show: graph T closed!